P 119

2. (a) $\int_0^1 (1+it)^2 dt$

$= \int_0^1 (1+2it-t^2)dt$

$= \left(t+it^2-\frac{t^3}{3}\right)\Big|_0^1$

$= \frac{2}{3}+i$

(b) $\int_1^2 \left(\frac{1}{t}-i\right)^2 dt$

$= \int_1^2 (t^{-2}-2it^{-1}-1)dt$

$= \left(-t^{-1}-2i\ln t-t\right)\Big|_1^2$

$= -\frac{1}{2}-i\ln 4$

(c) $\int_0^{\frac{\pi}{6}} e^{i2t} dt$

$= \int_0^{\frac{\pi}{6}} (\cos 2t + i\sin 2t)dt$

$= \frac{1}{2}(\sin 2t - i\cos 2t)\Big|_0^{\frac{\pi}{6}}$

$= \frac{1}{2}\left(\frac{\sqrt{3}}{2}-\frac{1}{2}i-(-1)i\right)$

$= \frac{\sqrt{3}}{4}+\frac{1}{4}i$

(d) $\int_0^\infty e^{-zt}\,dt$

$$= -\frac{1}{z}e^{-zt}\Big|_{t=0}^{t=\infty}$$

$$= -\frac{1}{z}(0-1)$$

$$= \frac{1}{z}$$

$\square$

4. $$\int_0^\pi e^{(1+i)x}\,dx = \left(\frac{1}{1+i}e^{(1+i)x}\right)\Big|_{x=0}^{x=\pi}$$

$$= \frac{1}{1+i}(-e^\pi - 1)$$

$$= \frac{1}{2}(1-i)(-e^\pi-1)$$

$$= -\frac{e^\pi+1}{2} + \frac{e^\pi+1}{2}i.$$

Hence, $\int_0^\pi e^x\cos x\,dx = -\frac{e^\pi+1}{2}$

$$\int_0^\pi e^x\sin x\,dx = \frac{e^\pi+1}{2}$$

$\square$

P124-125

6 (a). Let $z = x + iy(x)$.

Let $y(x) = 0$.

Then $x = 0$ or $\begin{cases} x^3 \sin(\frac{\pi}{x}) = 0 \\ 0 < x \le 1 \end{cases}$

i.e., $x = 0$ or $\begin{cases} \sin(\frac{\pi}{x}) = 0 \\ 0 < x \le 1 \end{cases}$

Then $x = 0$ or $x = \frac{1}{n}$, $n = 1, 2, 3, \cdots$

Hence $z = 0$ or $z = \frac{1}{n}$, $n = 1, 2, 3, \cdots$

(b) It suffices to check y is smooth near $x = 0$.

Since $0 \le \left| x^3 \sin(\frac{\pi}{x}) \right| \le x^3$ and $\lim_{x \to 0} x^3 = 0$,

then $\lim_{x \to 0} y(x) = 0 = y(0)$. Hence, y is continuous at $x = 0$

$$y'(0) = \lim_{x \to 0} \frac{y(x) - y(0)}{x} = \lim_{x \to 0} \frac{x^3 \sin(\frac{\pi}{x})}{x} = \lim_{x \to 0} x^2 \sin\frac{\pi}{x} = 0$$

(Again, by Squezze Theorem,

since $0 \le \left| x^2 \sin\frac{\pi}{x} \right| \le x^2$ and $\lim_{x \to 0} x^2 = 0$)

When $x > 0$, $y'(x) = 3x^2 \sin(\frac{\pi}{x}) - \pi x \cos(\frac{\pi}{x})$

Since $0 \le \left| x^2 \sin\frac{\pi}{x} \right| \le x^2$, $0 \le x \cos\frac{\pi}{x} \le x$ and

$\lim_{x \to 0} x^2 = \lim_{x \to 0} x = 0$, then $\lim_{x \to 0} y'(x) = 0 = y'(0)$

$\square$

P 132 - 135

1 (a) $$\int_C f(z)\,dz = \int_0^{\pi} \frac{2e^{i\theta}+2}{2e^{i\theta}} 2ie^{i\theta} d\theta$$

$$= 2i\int_0^{\pi} (e^{i\theta}+1)\, d\theta$$

$$= \left(2e^{i\theta} + 2i\theta\right)\Big|_0^{\pi}$$

$$= -2 + 2\pi i - 2$$

$$= -4 + 2\pi i$$

(b) $$\int_C f(z)\,dz = (2e^{i\theta} + 2i\theta)\Big|_{\pi}^{2\pi}$$

$$= 2 + 4\pi i - (-2) - 2\pi i$$

$$= 4 + 2\pi i$$

(c) $$\int_C f(z)\,dz = (-4 + 2\pi i) + (4 + 2\pi i)$$

$$= 4\pi i$$

$\square$

7. $$\int_C f(z)\,dz = \int_0^{\frac{\pi}{2}} (e^{i\theta})^{-1-2i} \cdot ie^{i\theta}\,d\theta$$

$$= i\int_0^{\frac{\pi}{2}} e^{2\theta}\,d\theta$$

$$= \frac{i}{2} e^{2\theta}\Big|_0^{\frac{\pi}{2}}$$

$$= i\,\frac{e^{\pi}-1}{2}$$

$\square$

9 (a) $$\int_C f(z)\,dz = \int_{-\pi}^{\pi} (e^{i\theta})^{-\frac{3}{4}} \cdot ie^{i\theta}\,d\theta$$

$$= i\int_{-\pi}^{\pi} e^{\frac{i}{4}\theta}\,d\theta$$

$$= 4e^{\frac{i}{4}\theta}\Big|_{-\pi}^{\pi}$$

$$= 4\left(\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i-\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i\right)$$

$$= 4\sqrt{2}\,i$$

(b) $$\int_C g(z)\,dz = 4e^{\frac{i}{4}\theta}\Big|_0^{2\pi}$$

$$= 4(i-1)$$

$$= -4+4i$$

$\square$

13. $$\int_C (z-z_0)^{n-1}\,dz = \int_{-\pi}^{\pi} (Re^{i\theta})^{n-1} Ri e^{i\theta}\,d\theta$$

$$= iR^n \int_{-\pi}^{\pi} e^{in\theta}\,d\theta$$

$$= \begin{cases} \frac{R^n}{n}\left(e^{n\pi i} - e^{-n\pi i}\right) = 0, & n = \pm 1, \pm 2, \cdots \\ 2\pi, & n = 0. \end{cases}$$

$\square$

P139

4. $$\left| \int_{C_R} \frac{2z^2-1}{z^4+5z^2+4}\,dz \right| \le \int_{C_R} \left| \frac{2z^2-1}{(z^2+1)(z^2+4)} \right| dz$$

$$\le \int_{C_R} \frac{2|z|^2+1}{(|z|^2-1)(|z|^2-4)}\,dz \quad \left(\begin{array}{l}\text{since } R>2 \\ \text{and } \Delta\text{-inequality}\end{array}\right)$$

$$= \int_{C_R} \frac{2R^2+1}{(R^2-1)(R^2-4)}\,dz$$

$$= \frac{\pi R(2R^2+1)}{(R^2-1)(R^2-4)}$$

$$= \frac{\pi\left(\frac{2}{R}+\frac{1}{R^2}\right)}{\left(1-\frac{1}{R^2}\right)\left(1-\frac{4}{R^2}\right)} \to 0 \quad \text{as } R \to \infty.$$

$\square$

5. $$\left|\int_{C_R} \frac{\text{Log}\, z}{z^2}\, dz\right| = \left|\int_{-\pi}^{\pi} \frac{\text{Log}(Re^{i\theta})}{(Re^{i\theta})^2} Rie^{i\theta} d\theta\right|$$

$$= \left|\int_{-\pi}^{\pi} \frac{i}{R} \frac{\ln R + i\theta}{e^{i\theta}}\, d\theta\right|$$

$$< \int_{-\pi}^{\pi} \frac{\ln R + \pi}{R}\, d\theta$$

$$= 2\pi\left(\frac{\pi + \ln R}{R}\right)$$

$$\lim_{R\to\infty} \frac{\pi + \ln R}{R} = \lim_{R\to\infty} \frac{1}{R} = 0$$

(by l'Hospital's rule)

$\square$

6. Since $f(z)$ is analytic and therefore continuous, $\exists\, M \geq 0$ s.t. $|f(z)| \leq M$.

$$\left|\int_{C_\rho} z^{-\frac{1}{2}} f(z)\, dz\right| \leq \int_{C_\rho} \left|z^{-\frac{1}{2}} f(z)\right| dz$$

$$= \int_{C_\rho} |z|^{-\frac{1}{2}} |f(z)|\, dz$$

$$\leq \int_{C_\rho} \frac{1}{\sqrt{\rho}} M\, dz$$

$$= 2\pi\rho \cdot \frac{1}{\sqrt{\rho}} M = 2\pi M\sqrt{\rho} \to 0 \text{ as } \rho \to 0.$$

$\square$

P 147

2. (a) $\int_0^{1+i} z^2 dz = \left.\frac{z^3}{3}\right|_0^{1+i}$

$= \frac{1}{3}(1+i)^3$

$= \frac{2}{3}(-1+i)$

(b) $\int_0^{\pi+2i} \cos(\frac{z}{2}) dz = \left.\left(2\sin\frac{z}{2}\right)\right|_0^{\pi+2i}$

$= 2\sin(\frac{\pi}{2}+i)$

$= -i(e^{(\frac{\pi}{2}+i)i} - e^{-(\frac{\pi}{2}+i)i})$

$= -i(ie^{-1}+ie)$

$= e + \frac{1}{e}$

(c) $\int_1^3 (z-2)^3 dz = \left.\frac{1}{4}(z-2)^4\right|_1^3$

$= 0$

$\square$

5. $$\int_{-1}^{1} z^i \, dz = \frac{1}{i+1} z^{i+1} \Big|_{-1}^{1}$$

$$= \frac{1}{i+1}\left(1 - \lim_{\theta \to \pi} e^{i\theta(i+1)}\right)$$

$$= \frac{1}{i+1}\left(1 - \lim_{\theta \to \pi} e^{-\theta} e^{i\theta}\right)$$

$$= \frac{1}{i+1}\left(1 - e^{-\pi} e^{i\pi}\right)$$

$$= \frac{1-i}{2}\left(1 + e^{-\pi}\right)$$

$\square$